ON THE AUTOMORPHISM GROUP OF A POSSIBLE SYMMETRIC $(81, 16, 3)$ DESIGN

A. ABDOLLAHI*, H. R. MAIMANI AND R. TORABI

ABSTRACT. In this paper we study the automorphism group of a possible symmetric $(81, 16, 3)$ design.

1. INTRODUCTION

Let v, k and λ be non-negative integers such that $v > k > \lambda$. By a symmetric (v, k, λ) design, we mean a pair $D = (V, \mathcal{B})$, where V is a v-set and $\mathcal{B}$ is a set of k-subsets of V such that the following four requirements are satisfied by D:

(1) $|\mathcal{B}| = v$.
(2) any element of V belongs to precisely k members of $\mathcal{B}$.
(3) any two distinct members of $\mathcal{B}$ intersect in exactly λ elements of V.
(4) any two distinct elements of V are in exactly λ members of $\mathcal{B}$.

As usual, the elements of V are called points of D and the members of $\mathcal{B}$ are called blocks of the design D. An automorphism of a symmetric design $D = (V, \mathcal{B})$ is a permutation on V which sends blocks to blocks. The set of all automorphisms of D with the composition rule of maps forms the full automorphism group of D which will be denoted by $Aut(D)$. If α is an automorphism

of D, we denote by $F(\alpha)$ the set of all points which are fixed by α; and $F_b(\alpha)$ denotes the set of all blocks which are fixed by α.

Over the years, researchers have tackled problems related to symmetric designs. The question of existence still remains unsettled for many parameter sets. Indeed, if we list the parameters (v, k, λ) in order of increasing $n = k - \lambda$, then $(81, 16, 3)$ would be the smallest unknown case [8]. On the other hand, the success of almost all the design construction methods depends heavily on a proper choice of possible automorphism groups [4].

As far as we know, the only known results on a possible $(81, 16, 3)$ design are the following:

Theorem 1.1. (See [2]) *There is no symmetric $(81, 16, 3)$ design with an abelian regular 3-group of automorphisms.*

Theorem 1.2. (See [7]) *Let α be an automorphism of a possible symmetric $(81, 16, 3)$ design of order 2. Then $|F(\alpha)| = 9$.*

Theorem 1.3. (See [5]) *The alternating group A_5 of degree 5 cannot be isomorphic to a group of automorphisms of a possible symmetric $(81, 16, 3)$ design.*

T. Spence has announced in his home page

http://www.maths.gla.ac.uk/ẽs/

that there is no symmetric $(81, 16, 3)$ designs having a "certain" fixed-point free automorphism of order 3.
Our main result is:

Theorem 1.4. *If G is the full automorphism group of a possible symmetric $(81, 16, 3)$ design, then $|G| = 2^\alpha 3^\beta 5^\gamma 13^\sigma$, where $\gamma \leq 1$, $\sigma \leq 1$. Moreover, G has no subgroup of order 65, and has no elements of orders 10 or 26; and G does not contain any abelian 2-subgroup of rank greater than 3.*

In Section 2, some general results on the automorphism groups of a symmetric design are given and in Section 3, we prove a series of Lemmata. Based on them we can prove Theorem 1.4.

2. Some general results on the automorphism group of a symmetric design

Lemma 2.1. (See [6]) *Let α be an automorphism of a nontrivial symmetric (v, k, λ) design. Then $|F(\alpha)| = |F_b(\alpha)|$.*

Lemma 2.2. (See [6, Corollary 3.7, p. 82]) *Let D be a non trivial symmetric (v, k, λ) design and α a non trivial automorphism of D. Then $|F(\alpha)| \leq k + \sqrt{k - \lambda}$.*

Lemma 2.3. *Let D be a symmetric (v, k, λ) design and α an automorphism of D of prime order p such that $\lambda < p$. If B is a block of D such that $|F(\alpha) \cap B| \geq 2$, then $B^\alpha = B$.*

Proof. Let x, y be two distinct elements of $F(\alpha) \cap B$. Then $x, y \in B = B^{\alpha^0}, B^\alpha, \ldots, B^{\alpha^\lambda}$. Since every two distinct points are in exactly λ blocks, $B^{\alpha^i} = B^{\alpha^j}$ for some distinct $i, j \in \{0, 1, \ldots, \lambda\}$. Thus $B^{\alpha^{i-j}} = B$. Since p is prime and $1 \leq |i - j| \leq \lambda < p$, $\gcd(i - j, p) = 1$. Therefore $B^\alpha = B$ as required. □

Lemma 2.4. *Let B_1 and B_2 be two distinct fixed blocks of the automorphism α of prime order p of a symmetric (v, k, λ) design with $\lambda < p$. Then $B_1 \cap B_2 \subseteq F(\alpha)$.*

Proof. Suppose, for a contradiction, that there exists a point $x \in (B_1 \cap B_2) \backslash F(\alpha)$. Thus $x^{\alpha^i} \neq x^{\alpha^j}$, for any two distinct $i, j \in \{0, 1, \ldots, p-1\}$; since otherwise $x^{\alpha^{i-j}} = x$ and so $x^\alpha = x$, as $\gcd(i - j, p) = 1$. It follows that $p = |\{x^\beta \mid \beta \in \langle \alpha \rangle\}|$. Since $B_i^\alpha = B_i$ for $i \in \{1, 2\}$, we have that $\{x^\beta \mid \beta \in \langle \alpha \rangle\} \subseteq B_1 \cap B_2$. Therefore $|B_1 \cap B_2| \geq p > \lambda$, a contradiction; since in symmetric (v, k, λ) designs, two distinct blocks intersect in exactly λ points. □

Lemma 2.5. *Let α be an automorphism of prime order p of a symmetric (v, k, λ) design with $\lambda < p$. Then*

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \leq v.$$

Proof. It follows from Lemma 2.4 that for any two distinct blocks B_1 and B_2 in $F_b(\alpha)$, $(B_1 \backslash F(\alpha)) \cap (B_2 \backslash F(\alpha)) = \varnothing$. This completes the proof. □

Lemma 2.6. *Let α be an automorphism of a symmetric (v, k, λ) design of prime order p such that $1 < \lambda < p$. Then $B \nsubseteq F(\alpha)$ for all blocks B.*

Proof. Suppose, for a contradiction, that there exists a block B such that $B \subseteq F(\alpha)$. Since every block $B_1 \neq B$ intersects B in $\lambda \geq 2$ points, it follows from Lemma 2.3 that every block is fixed under α. Thus $|F_b(\alpha)| = |F(\alpha)| = v$, by Lemma 2.1. Hence α is the identity automorphism; a contradiction. This completes the proof. $\square$

The following lemma is Theorem 2.7 of Aschbacher's paper [1].

Lemma 2.7. (Theorem 2.7 of [1]) *Let p be a prime divisor of the automorphism group of a symmetric (v, k, λ) design such that $1 < \lambda < p$ and $\gcd(p, v) = 1$. Then $p \leq k$.*

Proof. Suppose that α is an automorphism of the design of order p. Since α is a permutation on the point set, $F(\alpha) \equiv v \mod p$ and since $\gcd(p, v) = 1$, we have that $|F(\alpha)| \geq 1$. Thus, by Lemma 2.1, there exists a block B such that $B^\alpha = B$. Thus by Lemma 2.6, there exists an element $x \in B \backslash F(\alpha)$ and so $|\{x^\beta \mid \beta \in \langle \alpha \rangle\}| = p$. Since $B^\alpha = B$, we have that $\{x^\beta \mid \beta \in \langle \alpha \rangle\} \subseteq B$ and so $p \leq k$, as required. $\square$

3. Automorphism group of a possible symmetric (81,16,3) design

Lemma 3.1. *Let G be an automorphism group of a possible symmetric $(81, 16, 3)$ design which is elementary abelian 2-group. Then $|G| \leq 8$.*

Proof. Let r be the number of orbits of the action of G on the point set of the design. Then by the Cauchy-Frobenius Lemma (see [6, Proposition A.2, p. 246]),

$$r = \frac{1}{|G|} \sum_{\alpha \in G} |F(\alpha)|.$$

Since G is an elementary abelian 2-group, it follows from Theorem 1.2, that $|F(\alpha)| = 9$ for all non-identity elements α of G.

Let $|G| = 2^n$. Then, since $r = (2^n + 8) \cdot 9/2^n$ is an integer, we must have that 2^n divides $2^n + 8$ and so $n \leq 3$, as required. $\square$

Lemma 3.2. *Let G be an automorphism group of a possible symmetric $(81, 16, 3)$ design. Then G has no element of order 7 or 11.*

Proof. Suppose, for a contradiction, that G has an automorphism α of order p, where $p \in \{7, 11\}$. Since α is a permutation on a set with 81 elements, we have $|F(\alpha)| \equiv 81 \mod p$. Then it follows from Lemma 2.2 that

$$|F(\alpha)| \in \begin{cases} \{4, 11, 18\} & \text{if } p = 7 \\ \{4, 15\} & \text{if } p = 11 \end{cases}. \qquad (I)$$

Thus there are at least two distinct blocks which are fixed by α and so

$$|F(\alpha)| \geq 3 \qquad (*)$$

by Lemma 2.4. Now if $B \in F_b(\alpha)$, then α induces a permutation on the set B. Therefore $|F(\alpha) \cap B| \equiv 16 \mod p$ and so by $(*)$ and Lemma 2.6 we have

$$|F(\alpha) \cap B| = \begin{cases} 9 & \text{if } p = 7 \\ 5 & \text{if } p = 11 \end{cases}. \qquad (II)$$

If $p = 11$, then it follows from (I) and (II) that $|F(\alpha)| = 15$ and $|B \backslash F(\alpha)| = 11$ for all blocks $B \in F_b(\alpha)$; and if $p = 7$, then $|B \backslash F(\alpha)| = 7$ for all blocks $B \in F_b(\alpha)$ and $|F(\alpha)| \in \{11, 18\}$. Both cases contradict Lemma 2.5. This completes the proof. $\square$

Lemma 3.3. *Let α be an automorphism of a possible symmetric $(81, 16, 3)$ design of order 5. Then $|F(\alpha)| = 1$.*

Proof. Since α is a permutation on the point set, it follows from Lemma 2.2 that $|F(\alpha)| \in \{1, 6, 11, 16\}$. Suppose, for a contradiction, that $|F(\alpha)| \neq 1$. Let $B = B_1$ be an arbitrary block in $F_b(\alpha)$. Since $|F_b(\alpha)| = |F(\alpha)| \geq 2$, there exists a block $B_2 \neq B_1$ in $F_b(\alpha)$. By Lemma 2.4, $B_1 \cap B_2 \subseteq F(\alpha)$ and so there exist distinct elements x and y in $F(\alpha)$ which are both in B_1 and B_2. Therefore there exists a block B_3 distinct from B_1 and B_2 containing both x and y. Thus $3 = |B_i \cap B_j| \geq |B_1 \cap B_2 \cap B_3| \geq 2$ for

any two distinct $i, j \in \{1, 2, 3\}$. Now by Lemma 2.4, $B_i^\alpha = B_i$ for all $i \in \{1, 2, 3\}$ and so α is a permutation on B_i. Therefore, it follows from Lemmas 2.4 and 2.6, that $|F(\alpha) \cap B| \in \{6, 11\}$ for all blocks $B \in F_b(\alpha)$. Thus $|F(\alpha) \cap (B_1 \cup B_2 \cup B_3)| \geq 11$ and so $|F(\alpha)| \in \{11, 16\}$. If $|F(\alpha)| = 16$, then

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 16 + 16 \cdot 5 = 85,$$

which is a contradiction by Lemma 2.5. If $|F(\alpha)| = 11$, then there is no block $B \in F_b(\alpha)$ such that $|F(\alpha) \cap B| = 11$, since otherwise $|(B' \cup B) \cap F(\alpha)| \geq 11 + 6 - 3 = 14$ for any block $B' \in F(\alpha)$ distinct from B. Hence, in this case,

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 11 + 11 \cdot 10 = 121,$$

which contradicts Lemma 2.5. This completes the proof. □

Lemma 3.4. *Let G be an automorphism group of a possible symmetric $(81, 16, 3)$ design which is a 5-group. Then $|G| \leq 5$.*

Proof. It is enough to show that G has no subgroup H of order 5^2. If $\alpha \in G$ is of order 25, then by Lemma 3.3, $|F(\alpha)| = 1$, since $\varnothing \neq F(\alpha) \subseteq F(\alpha^5)$. Then, by Lemma 3.3, the number of orbits of the action of H on G is equal to

$$r = \frac{1}{5^2} \sum_{h \in H} |F(h)| = \frac{81 + 24 \cdot 1}{5^2} = \frac{21}{5}.$$

This is a contradiction, since r should be an integer. □

Lemma 3.5. *Let α be an automorphism of a possible symmetric $(81, 16, 3)$ design of order 13. Then $|F(\alpha)| = 3$.*

Proof. Since α is a permutation on the point set, it follows from Lemma 2.2 that $|F(\alpha)| \in \{3, 16\}$. Suppose, for a contradiction, that $|F(\alpha)| = |F_b(\alpha)| = 16$. Then, by Lemma 2.6, $|F(\alpha) \cap B| = 3$ for all $B \in F_b(\alpha)$. Thus

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \backslash F(\alpha)| \geq 16 + 16 \cdot 13 = 224,$$

contradicting Lemma 2.5. This completes the proof. □

Lemma 3.6. *Let G be an automorphism group of a possible symmetric $(81, 16, 3)$ design which is a 13-group. Then $|G| \le 13$.*

Proof. It is enough to show that G has no subgroup H of order 13^2. Since $13^2 > 81$, G has no element of order 13^2. Thus H is an elementary abelian 13-group. Then, by Lemma 3.5, the number of orbits of the action of H on G is equal to

$$r = \frac{1}{13^2} \sum_{h \in H} |F(h)| = \frac{81 + 12 \cdot 3}{13^2} = \frac{9}{13}.$$

This is a contradiction, since r should be an integer. $\square$

Lemma 3.7. *Let G be an automorphism group of a possible symmetric $(81, 16, 3)$ design. Then G has no element with the following orders: 10, 26, 65.*

Proof. (1) Suppose that G has an element of order 10. Then G contains two automorphisms α and β of orders 5 and 2 respectively such that $\alpha\beta = \beta\alpha$. Since α and β commutes, $\alpha(F(\beta)) = F(\beta)$. By Theorem 1.2 we have that $|F(\beta)| = 9$. Now by considering the cycle decomposition of α on $F(\beta)$, it follows that $|F(\alpha) \cap F(\beta)| \in \{4, 9\}$ which contradicts Lemma 3.3.

(2) Suppose that G has an element of order 26. Then G contains two automorphisms α and β of orders 13 and 2 respectively such that $\alpha\beta = \beta\alpha$. Since α and β commutes, $\alpha(F(\beta)) = F(\beta)$ and by Theorem 1.2, $|F(\beta)| = 9$, the cycle decomposition of α on $F(\beta)$ shows that $F(\beta) \subseteq F(\alpha)$ which contradicts Lemma 3.5.

(3) Suppose that G has an element of order 65. Then G contains two automorphisms α and β of orders 13 and 5 respectively such that $\alpha\beta = \beta\alpha$. Since α and β commutes, $\beta(F(\alpha)) = F(\alpha)$. But by Lemma 3.5 we have that $|F(\alpha)| = 3$ so the cycle decomposition of β on $F(\alpha)$ implies that $F(\alpha) \subseteq F(\beta)$ which contradicts Lemma 3.3. $\square$

Proof of Theorem 1.4. It follows from Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7

References

[1] M. Aschbacher, *On collineation groups of symmetric block designs*, J. Combinatorial Theory Ser. A **11**, 272–381 (1971).

[2] K. T. Arasu, $(81, 16, 3)$ *Abelian difference sets do not exists*, J. Combinatorial Theory Ser. A **43**, 350-343 (1986).

[3] A. Bowler, *On the fixed points of an automorphism of a symmetric design*, Discrete Math. **38**, 119-124 (1995).

[4] A. R. Camina, *A survey of the automorphism groups of block design*, J. Combin. Des. **2**, 79-100, (1994).

[5] M. Garapic, *The non-existence of symmetric design for* 2-$(81, 16, 3)$ *having A_5 as an automorphism group*, Glas. Math. Ser III **30(50)**, no.1, 3-8 (1995).

[6] E. Lander, *Symmetric Designs: an Algebraic Approach*, Cambridge University Press, 1983.

[7] Lj. Marangunic, *On involution acting on symmetric (81,16,3) designs*, Ars Combin. **4**, 261-269 (1995).

[8] Yury J. Ionin and Tran van Trung, *Symmetric Designs*, in: Handbook of combinatorial designs. Edited by Charles J. Colbourn and Jeffrey H. Dinitz. Second edition. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007.

Department of Mathematics, University of Isfahan, Isfahan 81746-71441, Iran; and Institute for Studies in Theoretical Physics and Mathematics (IPM); Tehran, Iran.

Deaprtment of Mathematics, University of Tehran, Tehran, Iran.

Deaprtment of Mathematics, University of Tehran, Tehran, Iran.

E-mail address: `a.abdollahi@math.ui.ac.ir`

E-mail address: `maimani@ipm.ir`